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STABILITY OF UNIFORM ROTATIONS OF A RIGID BODY ABOUT A PRINCIPAL AXIS

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A theorem on stability of steady motions of mechanical systems of specified type, is proved. The theorem is then used to investigate the stability of uniform rotations of a rigid body with a fixed point, about a principal axis containing the center of mass. We introduce an extended parametric space and define in this space a domain G of admissible parameter values. It is proved that uniform circular motions are stable in the subregion $G_1 \subset G$, in which the necessary conditions of stability are satisfied, except a certain set of dimension that is smaller by one than the dimension of the extended parametric space. A geometrical description of the domains G and G_1 is given for the case when two moments of inertia are equal.

The stability of the motions in question was studied by Rumiantsev [1], who obtained the sufficient conditions of stability using the Chetaev method to construct the Liapunov function in the form of a bundle of integrals of the equations of perturbed motion. As follows from [2-4], the sufficient conditions established in this manner, with the values of the parameters characterizing the rigid body being arbitrary, become necessary in the integrable cases of Euler, Lagrange and Kowalewska. In the nonintegrable cases, they no longer carry such a complete information about the character of the motion. It appears that the sufficient conditions of stability obtained in [1] become the necessary conditions only for the rotaions about the longest and the middle principal axis when the center of mass is below the support point. In the remaining cases these conditions either partly coincide, or the sufficient conditions are completely absent. This follows from the fact that in the neighborhood of the steady motions, the Hamiltonian of the reduced system needs not be a sign-definite function. Use of the Arnol'd's theorem (see [5]) to study a similar situation in the mechanical system with ignorable coordinates the reduced system of which is two-dimensional, makes it possible to prove a theorem on stability of steady motions of such systems. We use this theorem to extend considerably the region of stability of uniform rotations.

1. Stability of steady motions. Let us consider the steady motions of a mechanical system with $m + 2$ degrees of freedom and m ignorable coordinates. If canonical variables are used as the phase coordinates, then under the stability of a steady motion we shall understand, as usual, the Liapunov stability of this motion relative to all impulses and nonignorable coordinates q_1 and q_2. We can always assume that the steady motion in question corresponds to the point P with coordinates

$$q_1 = 0,\ q_2 = 0,\ p_1 = 0,\ p_2 = 0,\ p_{2+n} = c_n^\circ\ \ (n = 1, \ldots, m) \qquad (1.1)$$

Theorem 1. Let the Hamiltonian H be an analytic function of coordiantes and impulses at the point P, and let the Hamiltonian H° of the reduced system satisfy the following conditions at this point:

A. The eigenvalues of the linear reduced system are pure-imaginary and are $\pm i\alpha_1$ and $\pm i\alpha_2^\circ$.

B. The condition $k_1\alpha_1^\circ + k_2\alpha_2^\circ \neq 0$ holds for all integers k_1 and k_2 satisfying the inequality $|k_1| + |k_2| \leqslant 4$.

C. $D^\circ = -(\beta_{11}^\circ\alpha_2^{\circ 2} - 2\beta_{12}^\circ\alpha_1^\circ\alpha_2^\circ + \beta_{22}^\circ\alpha_1^{\circ 2}) \neq 0$, where $\beta_{\nu\mu}^\circ$ are the coefficients of the fourth order form of the Haniltonian H°, written in the following manner:

$$H^\circ = \sum_{\nu=1}^{2} \frac{\alpha_\nu^\circ}{2} R_\nu + \sum_{\nu,\mu=1}^{2} \frac{\beta_{\nu\mu}^\circ}{4} R_\nu R_\mu + O_5, \qquad R_\nu = \xi_\nu^2 + \eta_\nu^2$$

where O_5 is a power series with terms of at least fifth order. Then the steady motion (1.1) is Liapunov stable.

Proof. The proof of this theorem essentially depends on the sign of the product $\alpha_1^\circ\alpha_2^\circ$. When $\alpha_1^\circ\alpha_2^\circ > 0$, the Liapunov stability of the steady motion with a fixed $\mathbf{c}^\circ$ $(c_1^\circ, \ldots, c_m^\circ)$ follows from the positive definiteness of H_2. This enables us to apply the Routh theorem with the Liapunov complement [6], and to state that the motion is stable also when $\mathbf{c}^\circ$ is not fixed.

For $\alpha_1^\circ\alpha_2^\circ < 0$ we base the proof on the Mozer's proof [5] of the Arnol'd's theorem with the complement given in [7]. First we note that by virtue of the analytic character of H at the point P and of the condition $\alpha_1^\circ\alpha_2^\circ \neq 0$, the frequencies α_1 and α_2 are analytic functions of the cyclic constants $\mathbf{c}$ at point $\mathbf{c}^\circ$

$$\alpha_i = \alpha_i^\circ + \sum_{r=1}^{m} \frac{\partial\alpha_i}{\partial c_r}(c_r - c_r^\circ) + \frac{1}{2}\sum_{r,p=1}^{m} \frac{\partial^2\alpha_i}{\partial c_r\partial c_p}(c_r - c_r^\circ)(c_p - c_p^\circ) + \cdots, \qquad i = 1, 2 \tag{1.2}$$

(here and in the following the partial derivatives in the corresponding expansions are taken at the point $\mathbf{c}^\circ$). Therefore the conditions A and B of the theorem hold on the set

$$|\mathbf{c} - \mathbf{c}^\circ| \leqslant \varepsilon \tag{1.3}$$

where $|\mathbf{x}|$ is the Euclidean norm of the vector $\mathbf{x}$ and ε is a sufficiently small number. Consequently, for all $\mathbf{c}$ belonging to (1.3) there exists a Birkhoff transformation [8] which reduces the Hamiltonian H to the form

$$H = \sum_{\nu=1}^{2} \frac{\alpha_\nu}{2} R_\nu + \sum_{\nu,\mu=1}^{2} \frac{\beta_{\nu\mu}}{4} R_\nu R_\mu + O_5 \tag{1.4}$$

Since the functions defining this transformation are dependent analytically on the coefficients of the initial Hamiltonian, then $\beta_{\nu\mu}$ are analytic functions of $\mathbf{c}$ at the point $\mathbf{c}^\circ$

$$\beta_{\nu\mu} = \beta_{\nu\mu}^\circ = \sum_{r=1}^{m} \frac{\partial\beta_{\nu\mu}}{\partial c_r}(c_r - c_r^\circ) + \cdots, \qquad \nu, \mu = 1, 2 \tag{1.5}$$

and from the fact that the transformation is nondegenerate, it follows that the stability of the solution (1.1) is equivalent to the stability of the solution

$$\xi_1 = 0,\ \xi_2 = 0,\ \eta_1 = 0,\ \eta_2 = 0,\ p_{j+2} = c_j^\circ,\ j = 1, 2, \ldots, m \tag{1.6}$$

of the system

$$\frac{d\xi_i}{dt} = -\frac{\partial H}{\partial \eta_i}, \quad \frac{d\eta_i}{dt} = \frac{\partial H}{\partial \xi_i}, \quad \frac{dp_{j+2}}{dt} = 0, \qquad i = 1, 2, \; j = 1, 2, \ldots, m$$

where H is given by (1.4).

Let us assume that in the perturbed motion we have

$$\xi_i = \varepsilon x_i, \quad \eta_i = \varepsilon y_i, \quad p_{2+j} = c_j^\circ + \varepsilon c_j' \tag{1.7}$$

$$\mathbf{c}'^2 \leqslant 1 \tag{1.8}$$

Then the equations of perturbed motion assume the form

$$\frac{dx_i}{dt} = -\frac{\partial F}{\partial y_i}, \quad \frac{dy_i}{dt} = \frac{\partial F}{\partial x_i}, \quad \frac{dc_j'}{dt} = 0 \tag{1.9}$$

and admit the integrals

$$c_j' = \text{const}, \quad j = 1, 2, \ldots, m \tag{1.10}$$

$$F = \sum_{\nu=1}^{2} \frac{\alpha_\nu'}{2} R_\nu' + \varepsilon^2 \sum_{\nu,\mu=1}^{2} \frac{\beta_{\nu\mu}'}{4} R_\nu' R_\mu' + O(\varepsilon^3) = c \tag{1.11}$$

$$\alpha_\nu' = \alpha_\nu^\circ + \varepsilon \sum_{r=1}^{m} \frac{\partial \alpha_\nu}{\partial c_r} c_r + \frac{\varepsilon^2}{2} \sum_{r,p=1}^{m} \frac{\partial^2 \alpha_\nu}{\partial c_r \partial c_p} c_r c_p + \cdots$$

$$\beta_{\nu\mu}' = \beta_{\nu\mu}^\circ + \varepsilon \sum_{r=1}^{m} \frac{\partial \beta_{\nu\mu}}{\partial c_r} c_r + \cdots, \quad R_\nu' = x_\nu^2 + y_\nu^2$$

(the expressions for α_ν' and $\beta_{\nu\mu}'$ can be obtained from (1.2) and (1.5)). From now on we shall omit, for convenience, the primes. We note that

$$O(\varepsilon^3) < A\varepsilon^3 \tag{1.12}$$

for $0 < \varepsilon < A^{-1}$, where A is a certain independent of $\mathbf{c}$ and given by (1.8); this follows from the analyticity of the function H at the point P.

Let us study the behavior of the trajectories of the perturbed motion on the integral manifolds defined by a set of $m + 1$ constants C and $\mathbf{c}$. We shall show that on these manifolds the solutions $x(t)$ and $y(t)$ of the system (1.9) are uniformly bounded in C and c defined by the inequalities

$$|C| < |\alpha_1|/2, \quad c^2 \leqslant 1 \tag{1.13}$$

We now introduce the following new variables R_i and ϑ_i:

$$x_i = \sqrt{R_i} \sin \vartheta_i, \quad y_i = \sqrt{R_i} \cos \vartheta_i$$

Let us rewrite the differential equations in these variables

$$\frac{dR_\nu}{dt} = 2 \frac{\partial F}{\partial \vartheta_\nu} = O(\varepsilon^3) \tag{1.14}$$

$$\frac{d\vartheta_\nu}{dt} = -2 \frac{\partial F}{\partial R_\nu} = -2\left(\alpha_\nu + \varepsilon^2 \sum_{\mu=1}^{2} \beta_{\nu\mu} R_\mu\right) + O(\varepsilon^3)$$

Using $R_1 = R$ and $\vartheta_1 = \vartheta$, ϑ_2 is the independent variables we write, with the help of (1.11), the following expression for R_2:

$$R_2 = \Phi(R, \vartheta, \vartheta_2, C, \mathbf{c}) = -\frac{\alpha_1^\circ}{\alpha_2^\circ}\left(R - \frac{2C}{\alpha_1^\circ}\right) + \tag{1.15}$$
$$(A_1 + B_1 R)\varepsilon + \left(A_2 + B_2 R + \frac{D^\circ R^2}{2\alpha_2^{\circ 3}}\right)\varepsilon^2 + O(\varepsilon^3)$$

where A_1, A_2, B_1 and B_2 are functions of C and $\mathbf{c}$ bounded in (1.13). The inequality $A_1^2 + A_2^2 + B_1^2 + B_2^2 < M^2$, where M is a constant, holds in the region defined by (1.13), and the remainder term $O(\varepsilon^3)$ satisfies the estimate (1.12), where A is, in this case, independent of $\mathbf{c}$ and C from (1.13). In what follows, the properties of the transformations carried out ensure that the remainder terms $O(\varepsilon^3)$ will satisfy the same estimate in the region (1.13).

For any C from (1.13) and sufficiently small ε, the expression (1.15) is positive if $1 \leqslant R \leqslant 2$. Passing in (1.14) from t to ϑ_2, we find from (1.14) and (1.15)

$$\frac{dR}{d\vartheta_2} = \frac{\partial\Phi}{\partial\vartheta} = O(\varepsilon^3) \tag{1.16}$$
$$\frac{d\vartheta}{d\vartheta_2} = -\frac{\partial\Phi}{\partial R} = \frac{\alpha_1^\circ}{\alpha_2^\circ} - B_1\varepsilon - B_2\varepsilon^2 - \frac{D^\circ R}{\alpha_2^{\circ 3}}\varepsilon^2 + O(\varepsilon^3)$$

Let us integrate (1.16) with the accuracy of up to the terms of the order $O(\varepsilon^3)$

$$R(2\pi) = R(0) + O(\varepsilon^3) \tag{1.17}$$
$$\vartheta(2\pi) = \vartheta(0) + 2\pi\frac{\alpha_1^\circ}{\alpha_2^\circ} - 2\pi(B_1 + \varepsilon B_2)\varepsilon - \frac{2\pi\varepsilon^2}{\alpha_2^{\circ 3}} D^\circ R + O(\varepsilon^3)$$

When $D^\circ \neq 0$, the mapping (1.17) satisfies the conditions of the Mozer theorem [5], consequently an invariant curve Γ exists in the annulus $1 \leqslant R \leqslant 2$ on each integral manifold defined by C and $\mathbf{c}$ from (1.13). The remainder terms in (1.17) are uniformly bounded in C and $\mathbf{c}$ from (1.13), therefore $\varepsilon_0 > 0$ can be found independent of C and $\mathbf{c}$ and such, that for all $\varepsilon \in (0, \varepsilon_0)$ and C, $\mathbf{c}$ from (1.13) there exists an invariant curve Γ lying within the annulus $\varepsilon^2 \leqslant \eta_1^2 + \xi_1^2 \leqslant 2\varepsilon^2$. From this we conclude that if $\xi_1^2(0) + \eta_1^2(0) < \varepsilon^2$, then for any C and $\mathbf{c}$ from (1.13) we have

$$\xi_1^2(t) + \eta_1^2(t) \leqslant 2\varepsilon^2 \quad (t \geqslant 0) \tag{1.18}$$

The inequality (1.18) and (1.15) together yield the following estimate:

$$\xi_2^2(t) + \eta_2^2(t) < 3\left|\frac{\alpha_1^\circ}{\alpha_2^\circ}\right|\varepsilon^2 \quad (t \geqslant 0) \tag{1.19}$$

The inequalities (1.18) and (1.19) and the last relation of (1.7), together prove the stability of the solution (1.6), hence also that of (1.1), Q.E.D.

Notes. 1°. As follows from the proof, the requirement of analyticity of the Hamiltonian H at the point P which enables us to obtain the uniform upper bound for the remainder terms with respect to C and $\mathbf{c}$, can be replaced by another requirement of the existence, at the point P, of continuous fifth order partial derivatives in all arguments.

2°. If we assume that r components of the vector $\mathbf{c}$ are constructive parameters of the mechanical system and the remaining ones are, as before, cyclic constants, then Theorem 1 gives sufficient conditions of stability of the steady motions of a mechanical system with $m - r$ ignorable coordinates under the parametric perturbations [9] of constructive parameters.

2. Steady rotations of a body about its principal axis. The axes about which steady rotations are possible, form a Staude cone within the body [10]. By measuring out along each generatrix the value of the angular velocity with which the steady rotation takes place about the generatrix in question, we obtain the directrix. If the center of mass of the body lies on a principal axis, then one of the branches of the directrix coincides with this axis, and the body can rotate about this axis with any angular velocity. Let us investigate the stability of such motions relative to the projections of the angular velocity ω_1, ω_2 and ω_3 and of the vertical vector ν_1, ν_2 and ν_3 on the moving axes.

We shall use the Hamilton equations to describe the motion of the body. Juxtaposing the axes of the coordinate system associated with the body and the principal axes of the inertia ellipsoid, and introducing the Euler angles in the usual manner, we obtain the following expression for the Hamiltonian under the assumption that the center of mass lies on the first principal axis:

$$H = \frac{1}{2\sin^2\vartheta}\{a_1[(p_\psi - p_\varphi\cos\vartheta)\sin\varphi + p_\vartheta\cos\varphi\sin\vartheta]^2 + \tag{2.1}$$

$$a_2[(p_\psi - p_\varphi\cos\vartheta)\cos\varphi - p_\vartheta\sin\varphi\sin\vartheta]^2\} + \frac{a_3 p_\varphi^2}{2} + \Gamma\sin\varphi\sin\vartheta$$

Here a_1, a_2 and a_3 are the components of the gyration tensor, and Γ denotes the product of the weight of the body and the projection of the center of mass on the first axis.

A steady rotation at the angular velocity ω about the first principal axis is defined by the following values of the variables:

$$p_\vartheta = 0, \quad p_\varphi = 0, \quad p_\psi = \frac{\omega}{a_1}; \quad \vartheta = \frac{\pi}{2}, \quad \varphi = \frac{\pi}{2}, \quad \psi = \omega t + \psi_0 \tag{2.2}$$

The case $\Gamma > 0$ ($\Gamma < 0$) corresponds to the center of mass situated above (below) the point of suspension.

From (2.1) we see that a rigid body with a fixed point represents a mechanical system with three degrees of freedom and one ignorable coordinate. The steady motions of this system are uniform rotations of the body about the vertical. Investigation of the stability of the steady rotations with respect to ω_1, ω_2, ω_3, ν_1, ν_2 and ν_3 is equivalent to investigating the stability of the steady rotations with respect to p_ϑ, p_φ, p_ψ, ϑ and φ, consequently Theorem 1 is applicable to this problem. The steady motions in question are defined by (2.2) and the analysis which follows consists of investigating the Hamiltonian of the reduced system near these motions.

3. Expansion of the Hamiltonian near a uniform rotation. Assuming that

$$p_\vartheta = x_1', \quad p_\varphi = x_2', \quad \vartheta = \frac{\pi}{2} + y_1', \quad \varphi = \frac{\pi}{2} + y_2'$$

we find the expansion of the Hamiltonian of the reduced system near the position of equilibrium, with the accuracy of up to the fourth order terms in $x_1', \ldots, y_2'$

$$H = H_2 + H_4 + \ldots$$

$$2H_2 = a_2 x_1'^2 + a_3 x_2'^2 + (a_1 p_\psi^2 - \Gamma) y_1'^2 + [(a_2 - a_1) p_\psi^2 - \Gamma] y_2'^2 + 2(a_2 - a_1) p_\psi x_1' y_2' + 2a_1 p_\psi x_2' y_1'$$

$$2H_4 = (a_1 - a_2) x_1'^2 y_2'^2 + a_1 x_2'^2 y_1'^2 + \frac{8a_1 p_\psi^2 + \Gamma}{12} y_1'^4 +$$

$$\frac{4p_\psi^2(a_1-a_2)\Gamma}{12}y_2'^4+\frac{2(a_2-a_1)p_\psi^2+\Gamma}{2}y_1'^2y_2'^2+$$
$$\frac{4p_\psi(a_1-a_2)}{3}x_1'y_2'^3+(a_2-a_1)p_\psi x_1'y_1'^2y_2'+$$
$$\frac{5}{3}a_1p_\psi x_2'y_1'^3+2p_\psi(a_2-a_1)x_2'y_1'y_2'^2+2(a_2-a_1)x_1'x_2'y_1'y_2'$$

Let us pass to the dimensionless variables x_1, x_2, y_1 and y_2, and dimensionless time τ

$$(x_1', x_2')=\sqrt{|\Gamma|/a_1}(x_1, x_2),\quad (y_1', y_2')=(y_1, y_2),\quad \tau=t\sqrt{a_1|\Gamma|}$$

In the dimensionless form the equations of motion become

$$x_1^{\cdot}=-\frac{\partial H}{\partial y_1},\quad x_2^{\cdot}=-\frac{\partial H}{\partial y_2},\quad y_1^{\cdot}=\frac{\partial H}{\partial x_1},\quad y_2^{\cdot}=\frac{\partial H}{\partial x_2} \tag{3.1}$$

$$H=H_2+H_4+\cdots \tag{3.2}$$

$$2H_2=ax_1^2+bx_2^2+(\omega^2-e)y_1^2+[(a-1)\omega^2-e]y_2^2+$$
$$2(a-1)\omega x_1y_2+2\omega x_2y_1$$

$$2H_4=(1-a)x_1^2y_2^2+x_2^2y_1^2+\frac{8\omega^2+e}{12}y_1^4+\frac{4\omega^2(1-a)+e}{12}y_2^4+$$
$$\frac{2(a-1)\omega^2+e}{2}y_1^2y_2^2+\frac{4\omega(1-a)}{3}x_1y_2^3+(a-1)\omega x_1y_1^2y_2+$$
$$\frac{5}{3}\omega x_2y_1^3+2\omega(a-1)x_2y_1y_2^2+2(a-1)x_1x_2y_1y_2$$

$$a=\frac{a_2}{a_1},\quad b=\frac{a_3}{a_1},\quad \omega=p_\psi\sqrt{\frac{a_1}{|\Gamma|}},\quad e=\begin{cases}1, & \Gamma>0\\ -1, & \Gamma<0\end{cases}$$

where a dot ($\cdot$) denotes differentiation with respect to τ. The triangular inequalities for the moments of inertia define the domain C of variation of the parameters a and b. The domain C contains the positive values of a and b and is bounded by the curves $a=b(a+1)$, $b=a(b+1)$ and $a=b(a-1)$. It is depicted on Fig. 1.

4. Necessary conditions of stability. The characteristic equation of the linearized system with the function H_2 has the form

$$\lambda^4+Q_1\lambda^2+Q_2=0 \tag{4.1}$$
$$Q_1=(2+ab-a-b)\omega^2-e(a+b),\quad Q_2=[(a-1)\omega^2-$$
$$ea][(b-1)\omega^2-eb]$$

Therefore the necessary conditions of stability are

$$Q_1>0,\quad Q_2>0 \tag{4.2}$$
$$Q_1^2-4Q_2=(a+b-ab)^2\omega^4-2e(a+b-ab)(4-a-$$
$$b)\omega^2+(a-b)^2>0$$

The above conditions were obtained and analyzed thoroughly in [11]. Following [11] we also exclude from our discussion the critical cases in which some of the relations in (4.2) have the equal sign instead of the inequality sign.

We write the conditions of compatibility of the inequalities (4.2), using the notation adopted above. When $e = -1$, the inequalities (4.2) hold for any values of ω, for $b \geqslant a \geqslant 1$; if $b \geqslant 1 > a$ we have $\omega^2 < a/(1-a)$, while if $1 > a > b$ we have $\omega^2 < b/(1-b)$ or $\omega^2 > a/(1-a)$. When $e = 1$, we introduce for convenience the curves l_1 and l_2 defined by the equations

$$a = b(2b-3)/(b-1)^2,$$
$$b = (2a-3)/(a-1)^2$$

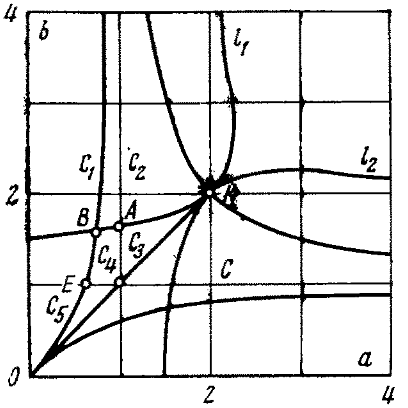

Fig. 1

respectively. These lines together with the straight lines $a = 1$, $b = 1$, $a = b$ divide the domain C into 10 subdomains C_i, C_i' ($i = 1, \ldots, 5$) (Fig. 1). The necessary conditions in the domains C_i' are obtained from the necessary conditions in C_i by replacing a by b and b by a in the corresponding inequalities. The domain C_1 includes the ray $a = 1$, $b \geqslant (\sqrt{5}+1)/2$ and the segment $[A, B]$ of the curve l_1, the domain C_2 includes the semi-interval $(A, K]$ of the same curve, C_4 includes the interval (A, D) and C_5 includes the semi-interval $[E, D)$. We note that on the line $a = b$ we have already established the sufficient conditions reflected by the Maievskii criterion, therefore the points lying on this line are not included in any of the domains C_i. Let us give the summary of necessary conditions of stability (the relevant domains are indicated in parentheses): the rotation is unstable for any value of ω (C_1);

$$\omega^2 > \frac{a}{a-1} \quad (C_2); \quad \omega_0^2 < \omega^2 < \frac{b}{b-1}, \quad \omega^2 > \frac{a}{a-1} \quad (C_3);$$

$$\omega_0^2 < \omega^2 < \frac{b}{b-1} \quad (C_4); \quad \omega^2 > \omega_0^2 \quad (C_5).$$

Here

$$\omega_0^2 = \frac{4-a-b+2\sqrt{(a-2)(b-2)}}{a+b-ab}$$

Notes. 1°. Rumiantsev in [1] used the Chetaev method to study the sufficient conditions of stability of the solution (2.2). These are found to be equivalent to the conditions of the sign-definiteness of H_2, consequently the problem of behavior of the solution (2.2) remains open in the following cases:

$$e = -1, \quad 1 > a > b, \quad \omega^2 > \frac{a}{1-a}$$

$$e = 1, \quad b > 1, \quad a > \frac{b(2b-3)}{(b-1)^2}, \quad \omega_0^2 < \omega^2 < \frac{b}{b-1}$$

and in the domains C_3, C_4 and C_5 in which the necessary conditions of stability hold, but the function H_2 has an alternating sign.

2°. The case $a = 1$ is considered separately below. In this case, the necessary conditions of stability are also sufficient when $e = -1$, while when $e = 1$, the sufficient conditions of stability cannot be obtained by constructing a Liapunov function for the integrals of the equations of perturbed motion since the function H^2 is of constant sign [12].

5. Reduction to normal form. In order to apply Theorem 1, we shall reduce the Hamiltonian (3.2) to its normal form, restricting ourselves to the terms of the fourth order inclusive. Denoting the roots of the Eq. (4.1) by $\pm i\alpha_1$ and $\pm i\alpha_2$, we write the following canonical transformation normalizing H_2 :

$$x_1 = s_1u_1 + c_1u_2, \quad y_1 = s_2v_1 + c_2v_2 \tag{5.1}$$

$$x_2 = s_3v_1 + c_3v_2, \quad y_2 = s_4u_1 + c_4u_2$$

$$s_1 = \alpha_1 [\alpha_1^2 + \omega^2 (a-1)(1-b) + be]\, w, \quad s_2 = [a\alpha_1^2 + \omega^2 (1-a)\, b + abe]\, w$$

$$s_3 = \omega\, [\alpha_1^2 (1-a) + \omega^2 (a-1) - ae]\, w, \quad s_4 = \alpha_1\omega\, (ab - a - b)\, w$$

$$\alpha_1 w^2 = c\, \{[\alpha_1^2 + \omega^2 (a-1)(1-b) + be][a\alpha_1^2 + \omega^2 (1-a)\, b + abe] + \omega^2 (ab - a - b)\, [\alpha_1^2 (a-1) + \omega^2 (1-a) + ae]\}^{-1}$$

Here c is an arbitrary constant. The formulas for c_1, c_2, c_3 and c_4 are obtained from the expressions for s_1, s_2, s_3 and s_4 in which α_1 are replaced by α_2. The coefficients β_{11}, β_{12} and β_{22} are equal to the coefficients accompanying $p_1^2q_1^2$, $p_1q_1p_2q_2$ and $p_2^2q_2^2$ in the form $4H_4$ written in terms of the complex variables p_1, p_2, q_1 and q_2 defined by

$$p_k = u_k + iv_k, \quad q_k = u_k - iv_k, \quad k = 1, 2$$

We obtain

$$8c\beta_{11} = 6(1-a)\, s_1^2s_4^2 + 6s_2^2s_3^2 + \frac{8\omega^2 + e}{2}\, s_2^4 + \frac{4\omega^2 (1-a) + e}{2}\, s_4^4 + [2(a-1)\,\omega^2 + e]\, s_2^2s_4^2 + 8\omega (1-a)\, s_1s_4^3 + 2(a-1)\,\omega s_1s_2^2s_4 + 10\omega s_2^3s_3 + 4\omega (a-1)\, s_2s_3s_4^2 + 4(a-1)\, s_1s_2s_3s_4$$

$$8c\beta_{12} = 2(1-a)\, [(c_1s_4 + s_1c_4)^2 + c_1c_4s_1s_4] + 2\, [(c_3s_2 + c_2s_3)^2 + c_2c_3s_2s_3] + (8\omega^2 + e)\, c_2^2s_2^2 + [4\omega^2 (1-a) + e]\, c_4^2s_4^2 + [2(a-1)\,\omega^2 + e]\, (c_2^2s_4^2 + s_2^2c_4^2) + 8\omega\, (1-a)\, (c_1s_4 + s_1c_4)\, c_4s_4 + 2\omega\, (a-1)(c_1c_4s_2^2 + s_1s_4c_2^2) + 10\omega c_2s_2\, (c_2s_3 + s_2c_3) + 4\omega\, (a-1)(c_2c_3s_4^2 + s_2s_3c_4^2) + 4(a-1)(c_2c_3s_1s_4 + s_2s_3c_1c_4)$$

The expression for β_{22} follows from the formula for β_{11} by replacing in the latter s_k by c_k.

6. The case when the moments of inertia are equal. Before analyzing the general case of the Hamiltonian of the reduced system, we turn our attention to the case $a = 1$ which corresponds to the equality $A_1 = A_2$ (A_1, A_2 and A_3 are the principal moments of inertia relative to the fixed point). Since the necessary conditions of stability become also sufficient when $e = -1$ (see Note 2°, Sect. 4), we shall assume from now on that $e = 1$. Then the coefficients β_{ke} become

$$16c\beta_{11} = 12s_2^2s_3^2 + 2s_2^2s_4^2 + 20\omega s_2^3s_3 + (8\omega^2 + 1)\, s_2^4 + s_4^4 \tag{6.1}$$

$$16c\beta_{22} = 12c_2^2c_3^2 + 2c_2^2c_4^2 + 20\omega c_2^3c_3 + (8\omega^2 + 1)\, c_4^2 + c_4^4$$

$$16c\beta_{12} = 2\,(4s_2s_3c_2c_3 + s_2{}^2c_3{}^2 + c_3{}^2s_3{}^2) + (8\omega^2 + 1)s_2{}^2c_2{}^2 + s_4{}^2c_4{}^2 + s_2{}^2c_4{}^2 + c_2{}^2s_4{}^2 + 10\omega s_2c_2\,(c_3s_2 + c_2s_3)$$

$$s_1 = \alpha_1\,(\alpha_1{}^2 + b)\,w, \quad s_2 = (\alpha_1{}^2 + b)\,w, \quad s_3 = -\,\omega w$$
$$s_4 = -\,\alpha_1\omega w$$

$$c_1 = \alpha_2\,(\alpha_2{}^2 + b)\,w', \quad c_2 = (\alpha_2{}^2 + b)\,w', \quad c_3 = -\,\omega w'$$
$$c_4 = -\,\alpha_2\omega w'$$

$$\alpha_1 w^2 = c\,[(\alpha_1{}^2 + b)^2 - \omega^2]^{-1}, \quad \alpha_2 w'^2 = c\,[(\alpha_2{}^2 + b)^2 - \omega^2]^{-1}$$
$$c = -\,16\omega^{-2}\alpha_1{}^2\alpha_2{}^2\,(\alpha_1{}^2 - \alpha_2{}^2)^2\,(\alpha_1{}^2 + b)\,(\alpha_2{}^2 + b)$$

Substituting the formulas (6.1) into the expression for the determinant D°, we obtain

$$D^\circ = (b-1)^2\omega^8 + 2\,(b-1)(b^2 + 2b - 5)\,\omega^6 + (b-1)(b^3 + 13b^2 - 41b + 7)\,\omega^4 + 8\,(b^4 - 5b^3 + 5b^2 + b + 2)\,\omega^2 + 4b\,(b-1)^2 \qquad (6.2)$$

From (6.2) we see that the condition C of Theorem 1 is violated only for the certain values of ω, namely for the roots of the equation $D^\circ = 0$. The points corresponding to these values of ω must be excluded from the domain of stability defined by the necessary conditions, since Theorem 1 does not provide a solution to the problem of stability of such steady rotations. Analyzing the equation $D^\circ = 0$ we conclude that it has no real roots when $1/2 \leqslant b \leqslant 1$ and has not more than four real roots when $1 < b \leqslant b^* = (1 + \sqrt{5})/2$. Figure 2 depicts the curve S_1 defined by the equation $D^\circ = 0$ in the $Ob\omega$-plane, for the interval $1/2 \leqslant b \leqslant (1+\sqrt{5})/2$ (all plots on Fig. 2 are constructed for $\omega > 0$, the lines for $\omega < 0$ are obtained by reflecting the above plots in the Ob-axis).

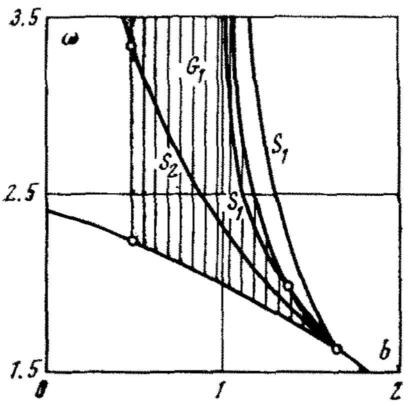

Fig. 2

Next we determine the steady rotations for which the condition B of Theorem 1 does not hold. The case $|\alpha_2| = |\alpha_1|$ was discussed in Sect. 4, therefore we shall not discuss it any further and choose the case $|\alpha_2| > |\alpha_1|$ to conclude that the following resonances may appear:

$$\alpha_2 = 2\alpha_1, \quad \alpha_3 = 3\alpha_1 \qquad (6.3)$$

The resonance $\alpha_2 = 2\alpha_1$ is not substantial since the expansion (3.2) of H contains no H_3-term. The last relation of (6.3) can be written after certain amount of manipulation, in the form

$$9\omega^4 + 2\,(41b - 59)\,\omega^2 + (9b - 1)(b - 9) = 0$$

and this yields a single positive value for ω^2

$$9\omega^2 = 59 - 41b + 10\sqrt{16b^2 - 41b + 34} \qquad (6.4)$$

which is always found to lie within the region of stability. Equation (6.4) defines a certain curve S_2 (Fig. 2) in the $Ob\omega$-plane.

To illustrate graphically the results obtained, we shall introduce an extended parametric space defined as a straight product of the parametric space of the mechanical system and of the space of cyclic constants. In the present case the $Ob\omega$-plane will serve as this space. The restrictions imposed on the moments of inertia separate, on this plane, a region $G\ (-\infty < \omega < \infty,\ \ ^1/_2 < b < \infty)$ of admissible parameter values. The region $(3 - b + 2\sqrt{2 - b} < \omega^2 < b\,/\,(b-1),\ \ ^1/_2 < b < (\sqrt{5} + 1)\,/\,2)$ in which the necessary conditions of stability hold, shall be denoted by G_1 (Fig. 2). Then the following theorem holds:

Theorem 2. Let a rigid body with equal moments of inertia about the first two axes rotate uniformly about the first axis which carries the center of mass situated above the point of suspension. Then the region of stability in the extended parameteric space, i.e. on the $Ob\omega$-plane, is represented by the region G_1 with the exclusion of the curves S_1 and S_2 (Fig. 2).

7. Regions of stability in the general case. Returning to the general case, we can use the results of Sect. 6 to assert that $D^\circ \not\equiv 0$. Then the equation D° $(a, b, \omega) = 0$ determines a certain surface S_1 in the extended parameteric space $Oab\omega$. Assuming that $|\alpha_2| > |\alpha_1|$, we find that the condition B of Theorem 1 is violated only when Eqs. (6.3) hold. As we said before, the resonance $\alpha_2 = 2\alpha_1$ is not substantial. From the analysis of the case $a = 1$ it follows that the resonance $\alpha_2 = 3\alpha_1$ is not fulfilled identically, therefore the equation $\alpha_2 = 3\alpha_1$ determines the surface S_2 in the parameteric space. Thus the conditions B and C of Theorem 1 fail only on the surfaces S_1 and S_2 in the space $Oab\omega$. Denoting, as before, the region of the extended parameteric space in which the necessary conditions of stability hold (see Sect. 4) by G_1, we can formulate the result obtained in the form of –

Theorem 3. Let a rigid body rotate uniformly about its first axis carrying the center of mass. Then the region of stability in the extended parameteric space $Oab\omega$ is represented by the region G_1 with the surfaces S_1 and S_2 exclude.

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A METHOD OF STUDYING THE STABILITY OF AUTONOMOUS SYSTEMS

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When Liapunov's direct method is used to study the stability of nonlinear systems and attempts are made to construct a Liapunov function with a derivative of constant sign or sign-definite, serious difficulties often occur. In the present paper a method is proposed for studying the stability of autonomous systems wherein use is made of an auxiliary function $V(\mathbf{x})$. The method is not connected with the conditions for $V(\mathbf{x})$ and its derivative with respect to time to be of constant sign or sign-definite. Instead, the function $V(\mathbf{x})$ along the trajectories of the system under study is required to satisfy a second order linear differential equation and certain boundary conditions. A theorem for the existence of the function $V(\mathbf{x})$ is proved and an effective method is given for constructing it is the solution of a Dirichlet problem for a degenerate elliptic operator of a special type; this makes it possible to obtain $V(\mathbf{x})$ numerically with the help of a computer. The function $V(\mathbf{x})$ can be used, not only for the study of stability, but also to determine regions of attraction and to obtain the invariant sets of autonomous systems, in particular, the limit cycles of second order systems.

1. We consider the system of equations of a perturbed motion

$$\mathbf{x}^{\cdot} = f(\mathbf{x}) \tag{1.1}$$

defined in some bounded domain $D \subset R^m$ and such that $f(\mathbf{x}) \in C^{(1)}(D)$. Here, and in what follows, by $C^{(k)}(D)$ we shall mean the space of functions which have in D continuous partial derivatives to order k inclusive, and by $C^{(k+\alpha)}(D)$ we shall mean the space of functions which have in D partial derivatives of order k which satisfy a Hölder condition with exponent $0 < \alpha < 1$. Let $\Omega = \{\mathbf{x} : \|\mathbf{x}\| \leqslant r\} \subset D$, and let Σ be the boundary of Ω. The intrinsic norm in R^m will be denoted by $\|\cdot\|$.

We introduce now an auxiliary system of equations for the perturbed motion

$$\mathbf{x}^{\cdot} = h(\mathbf{x}) \tag{1.2}$$

where